Exploiting Rank Deficiency for MR Image Reconstruction from Multiple Partial K-Space Scans

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ABSTRACT

In Magnetic Resonance Imaging (MRI) the acquired K-space data is corrupted by white Gaussian noise or by motion artifacts. In order to reduce the effects of these factors, it is a common practice to take multiple scans of the K-space. Noise/motion artifacts are reduced by averaging the K-space scans. For fully scanned K-space data, the image is obtained from this averaged K-space by applying the inverse Fourier transform. However, sampling the full K-space is time consuming. To reduce the scan-time smart reconstruction algorithms are employed obtain the MR image partial K-space scans. Generally Compressed Sensing (CS) based techniques are used to this end. In this work, we will show how the image can be reconstructed from multiple partial K-space scans by nuclear norm minimization. The reconstruction accuracy from our proposed method is the same as CS based techniques but is about ten times faster.

Index Terms— MRI, nuclear norm minimization

1. INTRODUCTION

Magnetic Resonance Images (MRI) is corrupted by noise and/or motion artifacts. It is assumed that these spurious effects are Normally distributed. To reduce these effects multiple K-space scans are obtained for each image. Signal averaging of the multiple K-space scans is done to reduce the effects of noise and/or artifacts. The image is obtained from the averaged K-space by applying 2D inverse Fourier transform.

MRI is a slow imaging technique; for the past two decades considerable effort has been directed in accelerating MRI scans. Signal averaging improves the Signal to Noise Ratio (SNR) of the output image but at the same time increases the acquisition time in proportion to the number of K-space scans collected before averaging.

The aim of this work is to reduce the MRI acquisition time but keep the reconstruction accuracy the same as traditional signal averaging. The only way to reduce the scan time is to partially sample the K-space instead of densely sampling it on a uniform Cartesian grid as is done conventionally. However, in such a situation it is not possible to obtain the image by applying 2D inverse Fourier transform; more sophisticated techniques are required.

In a very recent study [1], we have proposed a Compressed Sensing (CS) based reconstruction technique to obtain the image from multiple partial K-space scans. This is in line with the previous studies that proposed CS based techniques for reconstructing images from a single partially sampled K-space scan [2, 3].

CS based techniques exploit the sparsity of the MR image in a transform domain to reconstruct it from partially sampled K-space scan. In a recent study [4], we have shown that it is possible to reconstruct the MR image by exploiting its rank deficiency from partial K-space scan. The reconstruction accuracy is the same as CS based techniques but is about an order of magnitude faster. In this work, we propose to reconstruct the MR image from its multiple partially sampled K-space scans by exploiting its rank deficiency rather than its transform domain sparsity. The reconstruction accuracy is the same as our previous CS based technique [1] but is five to ten times faster.

The rest of the paper is organised into several sections. The next section gives a brief introduction to MR image reconstruction from partial K-space scans. Section 3, discusses in detail the problem and the proposed solution. The experimental results are shown in Section 4. The conclusions of the work are discussed in Section 5.

2. MR IMAGE RECONSTRUCTION FROM SINGLE K-SPACE SCAN

The data acquisition model for MRI can be expressed as:

$$y = Fx + \eta \qquad (1)$$

where y is the collected K-space samples, x is the underlying MR image (to be reconstructed), F is the Fourier mapping from the image space to the K-space and η is the noise.

To reduce the scan time, the K-space is not sampled fully. Thus the length of the vector y is smaller than x. Thus (1) represents an under-determined system of equation. To solve this inverse problem (1), prior information regarding the image to be reconstructed is required.

2.1 Exploiting Transform Domain Sparsity

MR images have a sparse representation in certain transform domains like wavelets or finite difference. By sparse, we mean that only a few coefficients are high valued while the rest are zeroes or very close to zero. Compressed Sensing (CS) based methods [2, 3] exploit the sparsity of the image in the transform domain to reconstruct it. In other words, transform domain sparsity is the a priori information utilized by CS methods to solve the under-determined inverse problem (1).

The image is solved via the following optimization problem:

$$\min \|Sx\|_1 \text{ subject to } \|y - Fx\|_2 \le \sigma \quad (2)$$

where σ is proportional to the standard deviation of noise. Here S is the sparsifying transform and σ is proportional to the standard deviation of noise. The l_1-norm over the transform domain coefficients promotes sparsity. In this work, we will use orthogonal wavelets as the sparsifying transform, but other transforms like finite differencing or redundant wavelets can be used as well.

2.2 Exploiting Rank Deficiency

In CS, the transform domain sparsity of the image is exploited to reconstruct the image from partial K-space scans. However, transform domain sparsity is not the only a priori information that can be utilized to reconstruct the image. MR images are rank deficient. In a recent work, we have shown that it is possible to reconstruct the image utilizing this information.

Ideally the image can be reconstructed by solving the following optimization problem,

$$\min rank(X) \text{ subject to } \|y - Fx\|_2 \le \sigma \quad (3)$$

where X is the image in matrix form.

However, this is an NP hard problem. Theoretical studies [5-7] have revealed that instead of solving the NP hard rank minimization problem, one can also attain the minimum rank solution by solving its nearest convex surrogate – the nuclear norm. Following these theoretical studies, we used the following convex optimization problem to reconstruct the MR image [4]:

$$\min \|X\|_* \text{ subject to } \|y - Fx\|_2 \le \sigma \quad (4)$$

where $\|X\|_*$ is the nuclear norm of the matrix defined as the sum of its singular values.

We found that the reconstruction accuracy using nuclear norm minimization (4) is almost the same as CS based method (2). However, our proposed method is about an order of magnitude faster.

3. THE PROBLEM AND THE PROPOSED SOLUTION

In order to improve the SNR, multiple K-space scans of the same cross section are acquired multiple times under the same condition. Mathematically this can be expressed as,

$$y_i = F_i x + \eta_i, i = 1...n \quad (5)$$

The different F_i's correspond to different K-space sampling patterns. For a full K-space scans all the F_i's are the same and equal the full FFT matrix.

In matrix vector notation, (5) can e expressed as,

$$\begin{bmatrix} y_1 \\ ... \\ y_n \end{bmatrix} = \begin{bmatrix} F_1 \\ ... \\ F_n \end{bmatrix} x + \begin{bmatrix} \eta_1 \\ ... \\ \eta_n \end{bmatrix}$$

In short, $\tilde{y} = \tilde{F}x + \tilde{\eta}$ (6)

We first discuss the traditional signal averaging technique. Then we will briefly discuss the CS based technique to solve the said problem. Finally, our proposed solution, based on nuclear norm minimization will be discussed in detail.

3.1 Signal Averaging

The MR signal acquired by the scanner is corrupted by white Gaussian noise. In order to improve SNR, multiple scans of the K-space are taken under similar under similar conditions. The usual practice is to average the data from the different scans in the K-space in view of reducing noise. The image is reconstructed by applying the 2D inverse Fourier transform.

Say, n K-space scans have been acquired. Now, signal averaging in the K-space leads to:

$$\frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n} Fx_i + \frac{1}{n}\sum_{i=1}^{n} \eta_i \quad (7)$$

The SNR improved image is obtained by taking the inverse 2D Fourier transform of the averaged K-space samples:

Since both averaging and Fourier transform are linear operations, it does not actually matter if one does the signal averaging first followed by the Fourier transform or vice versa. Therefore instead of following the traditional signal averaging followed by reconstruction one can first reconstruct the images for each K-space scan and then average the images and can expect to get the same results (this holds true because the noise is Gaussian the Fourier transform is an orthogonal transform).

Such averaging reduces the variance of the noise by *O*(n). This is an optimal approach, when the number of K-space scans is infinitely large.

3.2 CS based solution [4]

CS based techniques attempt to reconstruct the signal knowing only the partial samples of each of its K-space scans. A straightforward application of the traditional signal averaging technique implies that the same mask (R) is to be used for obtaining the partial K-space samples for every frame. After averaging the multiple K-space scans, CS reconstruction is carried out to reconstruct the underlying image. Mathematically the operation can be expressed as:

$$\min \|Sx\|_1 \text{ subject to } \left\| \frac{1}{n}\sum_{i=1}^{n} y_i - Fx \right\|_2 \le \frac{\sigma}{\sqrt{n}} \quad (8)$$

As this constitutes signal averaging followed by CS reconstruction, we denote it SA-CS.

This method is restrictive, in the sense that masks R's for all scans need to be the same. Alternately, one can perform CS reconstruction for the different K-space scans first and then average the resulting images. In such a case, it is not required that the masks be the same for all the scans. The CS reconstruction proceeds as,

$$\alpha_i = \min \|Sx\|_1 \text{ subject to } \|y_i - F_i x\|_2 \le \sigma \qquad (9)$$

The image is obtained by:

$$x = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad (10)$$

As in this method, CS reconstruction is done before signal averaging; we denote it by CS-SA.

The SA-CS and the CS-SA are naïve approaches to the said problem. In [4], we proposed a reconstruction technique that does away with signal averaging altogether. We formulated the reconstruction problem such that all the data from the multiple K-space scans are used by the optimization problem to output a single image.

The reconstruction can be carried out by straightforward application of l_1-norm minimization:

$$\min \|Sx\|_1 \text{ subject to } \|\tilde{y} - \tilde{F}x\|_2 \le \sigma \qquad (11)$$

3.3 Proposed Solution

We could formulate the nuclear-norm minimization equivalents of SA-CS and CS-SA methods. But in the previous work [4], we have argued that they are not optimal methods. Therefore, in this work, we formulate a recovery algorithm which data from all the partial K-space scans in a single optimization problem. This is a straightforward extension of (4) based on our discussion in the previous sub-section.

The nuclear norm minimization problem for recovering the MR image from multiple partial K-space scans will be:

$$\min \|X\|_* \text{ subject to } \|\tilde{y} - \tilde{F}x\|_2 \le \sigma \qquad (12)$$

Unlike CS, nuclear norm minimization is a very recent topic. Most of the studies are theoretical. An empirical work on rank deficient matrix recovery algorithms has been recently published by us [8]. In this work, we use the recovery algorithm as proposed in the aforesaid work.

4. EXPERIMENTAL EVALUATION

We wanted to ensure that the ground-truth is free from noise. Since practical MR images are always corrupted by noise, we worked only on synthetic data (Fig. 1). Both the images are of size 256 by 256 pixels. The K-space was simulated by taking the Fourier transform of the images and adding 10% white Gaussian noise to the real and complex parts separately. This was done five times in order to simulate K-space data corresponding to three scans.

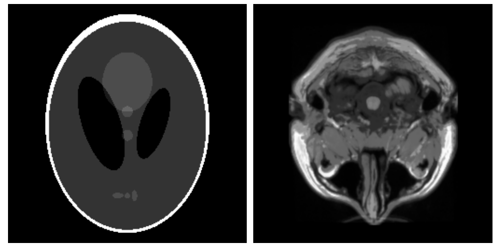

Fig.1. SheppLogan and Brainweb

For recovery via nuclear norm minimization, it is required that all the rows and the columns of the K-space be sampled. This is not possible when random lines in the frequency encoding or the phase encoding directions are omitted. Also random sampling does not guarantee that all the rows or columns are sampled at least once. To ensure sampling of all rows and columns efficiently, one has to resort to non-Cartesian sampling schemes like radial, spiral or rosetta. Radial sampling is one of the fastest sampling methods [9, 10] and has been used successfully for MR image reconstruction via nuclear norm minimization [4]; thus we use the same sampling technique in this work.

The l_1 minimization problem (11) was solved by the Spectral Projected Gradient (SPGL1) algorithm [11]. For solving the nuclear norm minimization problem, we used the shrinkage algorithm developed in [8] and successfully used for practical problems in [4] (MR image reconstruction) and [12] (sensor networks).

Normalized Mean Squared Error (NMSE) is used as the metric for comparing the results. The numerical results for different sampling ratios – 20% and 40% are shown in Table 1.

Table 1. Reconstruction accuracy in NMSE

Image	Method	20% sampling	40% sampling
SheppLogan	CS recovery	0.117	0.041
	Proposed	0.124	0.048
Brainweb	CS recovery	0.051	0.033
	Proposed	0.046	0.030

Traditional signal averaging technique yields a reconstruction accuracy of 0.4 and 0.3 for the SheppLogan and Brainweb images respectively. Comparing these figures with Table 1 concludes that both the CS based and our proposed methods are as accurate as traditional signal averaging, but requires only 40% samples. The reconstruction accuracy from can be seen qualitatively from the reconstructed images shown in Fig. 2.

The reconstruction times for the proposed method and the CS based technique is shown Table 2.

Table 2. Reconstruction times in seconds

Image	Method	20% sampling	40% sampling
SheppLogan	CS recovery	120	202
	Proposed	10	32
Brainweb	CS recovery	82	130
	Proposed	10	22

The SPGL1 is one of the fastest l_1-minimization algorithms in CS literature. We see that the reconstruction accuracy from our proposed method is about an order of magnitude faster than one of the fastest known CS reconstruction algorithm.

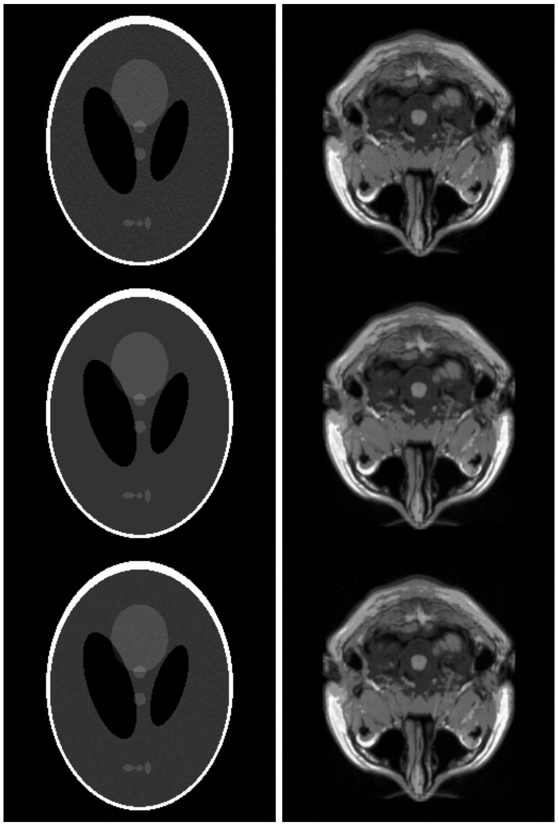

Fig. 1. From top to bottom –CS reconstruction, Proposed and signal averaging on full K-space.

5. CONCLUSION

To acquire MR images with high signal to noise ratio a usual practice is to acquire multiple K-space scans of the same cross section. Signal averaging is used to reduce the noise in the K-space by taking a mean of all the scans. When the full K-space data is available, 2D inverse Fourier transform is applied to obtain the image.

This technique improves the SNR but acquiring multiple K-space scans is time consuming. Recently a Compressed Sensing (CS) based method was proposed to reconstruct images from multiple partial K-space scans [4].

In this paper, we showed that instead of exploiting the transform domain sparsity of the underlying MR image (as done by CS) similar reconstruction accuracy can be obtained by utilizing the information that the images are rank deficient. This leads to a recovery algorithm which requires minimizing the nuclear norm of the image matrix. As mentioned earlier, our proposed technique is as accurate as the CS based technique but is about an order of magnitude faster.

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