Exploiting Sparsity and Rank-deficiency in Dynamic MRI Reconstruction

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ABSTRACT

This work addresses the problem of dynamic MRI reconstruction from partially sampled K-space. When the frames of the dynamic MRI sequences are stacked as columns of a matrix, the resultant matrix is both sparse (in a transform domain) and rank-deficient. The dynamic MRI sequence is reconstructed by solving an optimization problem that minimizes a sum of sparsity and rank-deficiency penalties subject to data constraints (K-space data acquisition model). In this work, we propose a non-convex optimization problem for dynamic MRI reconstruction where the sparsity penalty is an l_p-norm and the rank-deficiency penalty is the Schatten-q norm ($0<p,q\leq 1$). There is no algorithm to solve this combined l_p-norm and Schatten-q norm minimization problem; hence we derive a new algorithm based on the Majorization Minimization method. Our proposed method shows considerable improvement in reconstruction results over state-of-the-art techniques in dynamic MRI reconstruction.

***Index Terms*—** MRI, Sparsity, Rank-deficiency.

1. INTRODUCTION

In Magnetic Resonance Imaging (MRI) the data is acquired in the Fourier frequency domain which is traditionally called the 'K-space' in MRI literature. When the full K-space is sampled uniformly on a Cartesian grid, the reconstruction is trivial – applying an inverse FFT to produce the image. However sampling the full K-space on a uniform grid is time consuming and poses a problem for both static and dynamic MRI.

In dynamic MRI, one is ideally interested in acquiring a sequence of images of high spatial and temporal resolution. Unfortunately one comes at the cost of other – high spatial resolution compromises the frame-rate and vice versa. One way to increase the temporal resolution without sacrificing the spatial resolution is to partially sample the K-space and use a smart reconstruction technique that exploits prior knowledge about the dynamic MRI sequence.

Before discussing the reconstruction techniques for under-sampled data, we will briefly describe the dynamic MRI K-space data acquisition model. Let X_t denote the MR image frame at the t^{th} instant and T the total number of frames. Let y_t be the acquired K-space data for the t^{th} frame. The problem is to recover all X_t's ($t=1\ldots T$) from the collected k-space data y_t's. The MR imaging equation for each frame is as follows,

$$y_t = RFx_t + \eta_t \text{ where } t = 1 \ldots T \quad (1)$$

where F is the Fourier transform matrix/operator (2D or 3D as the case may be) which maps the image space to the k-space, R is the under-sampling mask applied on the k-space, x_t is the vector form of the MR image X_t to be recovered and η_t is the noise assumed to be distributed Normally.

The data acquisition model (1) can be expressed as,

$$Y = RFX + \eta \quad (2)$$

where $Y = [y_1|\ldots|y_T]$, $X = [x_1|\ldots|x_T]$, $\eta = [\eta_1|\ldots|\eta_T]$ and T is the total number of frames .

The problem is to recover the dynamic MRI sequence X given the K-space samples Y. Since, the K-space is partially sampled the problem (2) is under-determined and therefore does not have a unique solution. To get a practical solution to (2), one needs to have a prior knowledge about X.

In a dynamic MRI sequence the difference between frames arises from motion (heart beat) or from changes in concentration (cardiac perfusion). The motion/concentration change is typically concentrated only in certain areas of the cross section under study. Thus along the temporal direction of the sequence, (along rows of X), only certain areas (corresponding to motion/change in concentration) have major variations in pixel values while the rest of the areas have negligible variation.

Under such an assumption on the dynamic MRI sequence, the signal X will be approximately sparse under several transforms –

1. When the 1D Fourier transform is applied along the temporal direction [1], the resultant signal in the x-f[1] space will be approximately sparse since most areas show small variation in pixel values and will lead to Fourier transform coefficients near to zero; only the small areas where the variation is high will result in high valued coefficients.

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[1] The signal X is said to be in the x-t space, where x refers to spatial domain and t refers to temporal direction. When 1D Fourier transform is applied on X in the temporal direction, the resultant signal is in the x-f space where f refers to temporal frequency.

2. When temporal differencing is applied on X [2], the resultant difference signal is also approximately sparse since most of the areas with low time variations cancel each other (leading to zeroes after differencing) while only small areas showing motion/change in concentration correspond to high values in the resulting difference signal.
3. When spatio-temporal differencing is applied on X [3], the resultant signal is also sparse. Spatio-temporal differencing additionally assumes that the images at each time instant are piece-wise smooth.

There are a few papers [4, 5] that assume the signal X to be rank deficient. Since the columns of X (i.e. image frames) are correlated, the signal X can be expressed by very few temporal basis functions and hence it is approximately rank deficient.

A new method called '*k-t* SLR' [6] proposes minimization of a combination of rank-deficiency of the signal in the *x-t* space and its sparsity in Spatio-Temporal differencing domain. The following optimization problem is proposed,

$$\min_X \|Y - RFX\|_F^2 + \lambda_1 \|\nabla_s \nabla_t (x)\|_1 + \lambda_2 \|X\|_{S_q}^q \quad (3)$$

where x is the vectorised version of X, ∇_s is the finite differencing operator in the spatial domain, ∇_t is the temporal differencing operator and λ_1, λ_2 are the Lagrange multipliers; $\|.\|_F$ is the Frobenius norm of the matrix, $\|.\|_1$ is the l_1-norm of the vector and $\|.\|_{S_q}^q$ is the Schatten-q norm of the matrix raised to the power q.

Here $\|Y - RFX\|_F^2$ is the data fidelity term, $\|\nabla_s \nabla_t (x)\|_1$ is the convex sparsity penalty and the non-convex Schatten-q norm $\|X\|_{S_q}^q$ is the penalty on rank-deficiency. The constants λ_1 and λ_2 control the relative importance of the sparsity and rank-deficiency penalties. The idea of combining sparsity with rank-deficiency for dynamic MRI reconstruction was first proposed in *k-t* SLR [6]. The k-t SLR method yields better results than other reconstruction techniques that rely only on sparsity or only on rank-deficiency but does not combine both.

Following the success of k-t SLR a recent work proposed a variation of it [7]. The fundamental idea remains the same (combining sparsity with rank-deficiency) but it differs from k-t SLR in two aspects –

1. Instead of assuming the signal to be sparse in the spatio-temporal differencing domain, it was assumed to be sparse in the *x-f* space.
2. Instead of using Schatten-q norm as the penalty on rank-deficiency, a power factorization based method was used to account for rank-deficiency.

In this paper, we adhere to the same fundamental assumption as in [6, 7]. For the sparsity penalty, we will assume that the signal X is sparse in the *x-f* space as in [1, 7]. But instead of using the convex l_1-norm as sparsity penalty as used in [1-3, 6, 7] we propose using the non-convex l_p-norm. We replace the convex l_1-norm by the non-convex l_p -norm as the sparsity penalty because it has been shown in previous studies [8-10] to yield better MRI reconstruction results can be achieved thus. For the rank-deficiency penalty we use the Schatten-q norm as used in [6]. Thus, we propose to reconstruct the dynamic MRI sequence X by solving the following optimization problem,

$$\min_X \|Y - RFX\|_F^2 + \lambda_1 \|F_{1D} X^T\|_p^p + \lambda_2 \|X^T\|_{S_q}^q \quad (4)$$

Here $F_{1D} X^T$ transforms the signal X from the *x-t* space to the *x-f* space. The l_p-norm is defined over the vectorised version of $F_{1D} X^T$. The values of p and q lie between 0 and 1. Instead of imposing the Schatten-q norm on X as in (3) we define it on X^T; this is a trivial change but will help us later to simplify the problem.

Unfortunately there is no algorithm to solve (4). We derive an efficient algorithm to solve it based on the Majorization Minimization approach [11]. The derivation of the algorithm is given in the next section. In section 3, the experimental results are described. Finally the conclusions of the work are discussed in section 4.

2. ALGORITHM DERIVATION

The problem is to solve (4). Instead of solving it directly we will first simplify it. We replace the Schatten-q norm by its equivalent Ky-Fan norm ($\|U\|_{S_q}^q = Tr(U^T U)^{q/2}$) and substitute $Z = F_{1D} X^T$. Using these substitutions (4) takes the simplified form,

$$\min_Z \|Y - RFZF_{1D}\|_F^2 + \lambda_1 \|Z\|_p^p + \lambda_2 Tr(Z^T Z)^{q/2} \quad (5)$$

Since $\|F_{1D}^T Z\|_{S_q}^q = Tr(Z^T F_{1D} F_{1D}^T Z)^{q/2} = Tr(Z^T Z)^{q/2}$.

Using the Kronecker product notation, the term $\|Y - RFZF_{1D}\|_F^2$ in (5) can be expressed as, $\|y - (F_{1D}^T \otimes RF) z\|_2^2$ where $z = vec(Z)$ and $y = vec(Y)$, i.e. the oprator 'vec' converts a matrix to a vector by row/column concatenation.

Using these substitutions, the simplified optimization problem that needs to be solved is,

$$\min_z \|y - Az\|_2^2 + \lambda_1 \|z\|_p^p + \lambda_2 Tr(Z^T Z)^{q/2} \quad (6)$$

where $A = F_{1D}^T \otimes RF$ and $z = vec(Z)$.

We solve this problem by the Majorization-Minimization (MM) approach [11]. The generic MM algorithm is as follows,

Let $J(x)$ be the (scalar) function to be minimized

1. Set iteration count $k=0$ and initialize x_0.

Repeat step 2-4 until a suitable stopping criterion is met.

2. Choose $G_k(x)$ such that
   a. $G_k(x) \geq J(x)$ for all x.
   b. $G_k(x_k) = J(x_k)$.
3. Set x_{k+1} as the minimizer for $G_k(x)$.
4. Set k=k+1, go to step 2.

For our problem the function to be minimized is

$$J(x) = \|y - Ax\|_2^2 + \lambda_1 \|x\|_p^p + \lambda_2 Tr(X^T X)^{q/2}$$

There is no closed form solution to $J(x)$, it must be solved iteratively. At each iteration we choose,

$$G_k(x) = \| y - Ax \|_2^2 + (x - x_k)^t (aI - A^T A)(x - x_k) \\ + \lambda_1 \|x\|_p^p + \lambda_2 Tr(X^T X)^{q/2} \quad (7)$$

$G_k(x)$ satisfies the condition for MM algorithm when $a \geq \max eigvalue(A^T A)$

Now $G_k(x)$ can be alternately expressed as follows,

$$G_k(x) = a \| x_k + \frac{1}{a} A^T (y - Ax) - x \|_2^2 \\ + \lambda_1 \|x\|_p^p + \lambda_2 Tr(X^T X)^{q/2} + K \quad (8)$$

where K consists of terms independent of x.

Minimizing (8) is the same as minimizing the following,

$$G_k'(x) = \| b - x \|_2^2 + \frac{\lambda_1}{a} \|x\|_p^p + \frac{\lambda_2}{a} Tr(X^T X)^{q/2} \quad (9)$$

where $b = x_k + \frac{1}{a} A^T (y - Ax_k)$.

The problem now is to minimize (9). To do so, we take the derivative of the function,

$$G_k'(x) = 2x - 2b + \frac{\lambda_1}{a} p |x|^{p-2} .* x + \frac{\lambda_2}{2a} q (XX^T)^{\frac{q}{2}-1} X \quad (10)$$

where '.' denotes element wise product.

Setting the gradient to zero, one gets,

$$(I + D)x = b \quad (11)$$

where $D = \frac{\lambda_1}{2a} p Diag(|x|^{p-2}) + \frac{\lambda_2}{4a} qI \otimes (XX^T)^{\frac{q}{2}-1}$.

Here the Diag operator creates a diagonal matrix out of the vector $|x|^{p-2}$.

The problem (11) represents a system of linear equations. It should be noted that the system $(I+D)$ is symmetric. Hence it can be efficiently solved by newly developed MINRES-QLP algorithm [12].

Based on this derivation, we propose the following algorithm to solve (6).

Intitialize: $x_0 = 0$

Repeat until: $\|y - Ax\|_2^2 \leq \varepsilon$

Step 1. $b = x_k + \frac{1}{a} A^T (y - Ax_k)$

Step 2. $D = \frac{\lambda_1}{2a} p Diag(|x_{k-1}|^{p-2}) + \frac{\lambda_2}{4a} qI \otimes (X_{k-1} X_{k-1}^T)^{\frac{q}{2}-1}$

Step 3. Compute x by solving $(I + D)x = b$

End

The stopping criterion here is based on the data consistency term. The iterations stop when $\|y - Ax\|_2^2 \leq \varepsilon$. Here ε is defined as the product of the number of pixels in the image frames times the number of dynamic MRI frames collected (length of the sequence) times the variance of noise (η).

3. EXPERIMENTAL RESULTS

In this work, we compare our method with two state-of-the-art methods in dynamic MRI reconstruction, that are based on exploiting the transform domain sparsity and the rank-deficiency of the image sequence [6, 7]. In [6], the sparsity penalty is the l_1-norm on the Spatio-temporal TV (total variation) and the rank-deficiency penalty is on the Schatten-q norm; in [7] the sparsity penalty is the l_1-norm on the *x-f* space and the rank-deficiency is exploited via a power factorization based method.

All the methods require specification of the two parameters λ_1 and λ_2 – they control the relative importance of the sparsity and the rank-deficiency penalties respectively. Unfortunately these parameters cannot be determined based on rigorous optimization theory. They need to be tuned. The tuning mechanisms for determining these values are not clearly mentioned in [6, 7]. Therefore in this work, we follow a tuning methodology outlined in [3]. The value of λ_1 was fixed by using the L-curve method after putting λ_2 to zero, thereby using sparsity only. Once the value of λ_1 is fixed, the value of λ_2 is then chosen by minimizing the error in the reconstruction, as compared with ground-truth. For our algorithm, we also need to specify the value of ε; in this work it is assumed that the K-space data is not corrupted by noise hence ε is fixed at a small value of 10^{-3}.

In [6] a value of $q = 0.1$ is used for the Schatten-q norm. We use the same value of q for our proposed method. Our method also requires specifying the value of p in the l_p-norm; we use $p = 0.1$. We found experimentally that the results are not very sensitive to the value of p as long as it varies between *0.1* and *0.4*.

The experiments were carried out on three different datasets (Fig. 1). The first and second datasets were comprised of the 'Larynx' and the 'Cardiac' sequences respectively. The data were obtained from [13]. The larynx sequence was of size 256 x 256, the cardiac sequence was of size 128 x 128 for each time frame and 6 images were collected per second. The third dataset was obtained from [14]. It consisted of a dynamic MRI scan of a person repeating the word 'elgar'. This is the 'Speech' Sequence [14]. The image was of resolution 180 x 180 and was obtained at the rate of 6 frames per second.

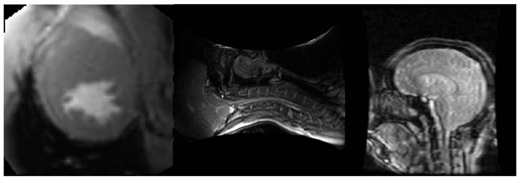

Fig. 1. Left to Right: Cardiac. Larynx and Speech

For all the data sequences, radial sampling was used. This is a non-Cartesian sampling scheme. The mapping from the Cartesian image space to the Non-Cartesian Fourier space is obtained via the Non-Uniform FFT [15]. We provide the results for 48 radial lines.

In this work we measure the quantitative reconstruction accuracy in terms of the signal to noise ratio (SNR). However, SNR does not provide a qualitative measure on the reconstruction accuracy, therefore we also provide the difference images (difference between the reconstructed and the ground-truth) for qualitative evaluation.

Table 1. Reconstruction Accuracy (SNR) for 48 radial lines

Sequence Name	[6]	[7]	Proposed
Larynx	22.6	21.9	23.8
Cardiac	22.9	22.4	24.6
Speech	15.2	14.8	16.8

The tables show that our proposed method yield the best reconstruction results. In order to test if our reconstruction results show a significant improvement over the previous methods we carried out a simple statistical t-test. We found that the reconstruction accuracy from our method is significantly different (better) than [6] and [7]. However our tests revealed that the reconstruction results from [6] and [7] are not significantly different from each other. The tests were carried out at 5% confidence interval.

For the qualitative aspects of the reconstruction, we provide the reconstructed and difference images for the Speech sequence. Owing to limitations in space we can show the images for only one sequence. The contrast of difference images have been magnified 5 times for visual clarity.

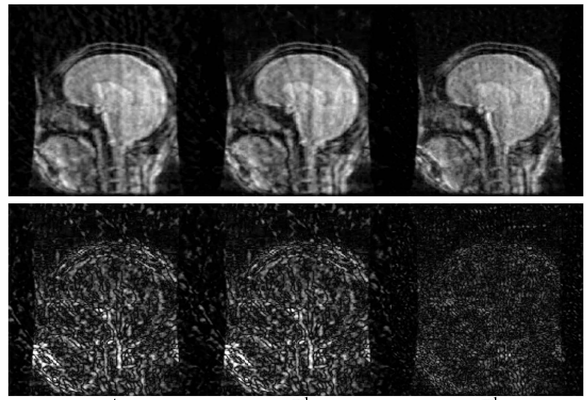

Fig. 2. 1st Column: [6]; 2nd Column: [7]; 3rd Column: Proposed

Our proposed method gives the best reconstruction results. The difference image is the darkest, meaning that the error between the ground-truth and the reconstructed image is the least. The difference images from [6] and [7] are almost the same and worse than our proposed method.

4. CONCLUSION

In this work we propose a new method for dynamic MRI reconstruction. Dynamic MRI sequences are temporally correlated. When the MRI frames are stacked as columns of a matrix, the resultant matrix has a sparse representation in the *x-f* space and also is rank deficient. This information was exploited previously in [6, 7] to yield good dynamic MRI reconstruction results. Our work follows from [6, 7] but deviates from them in one key aspect. We use non-convex penalties both for sparsity and rank-deficiency. Experimental results show that our method yields significantly better results than [6, 7].

Our method requires solving a least squares optimization problem regularized with an l_p-norm (as the sparsity penalty) and Sa chatten-q norm (as the rank-deficiency penalty). There is no efficient algorithm to solve this problem. We derived an efficient algorithm to solve it based on the Majorization Minimization approach.

In this work, we assume that the dynamic MRI sequence is temporally correlated which gives rise to the fact that it is sparse in *x-f* space. It does not take into account the information that the MR image (at each time point in the sequence) is also spatially correlated. In future, we intend to exploit this information as well, leading to full utilization of the spatio-temporal redundancy in the dynamic MRI sequence. This is likely to improve the results even more.

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