A FOCUSS BASED METHOD FOR LOW RANK MATRIX RECOVERY

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ABSTRACT

In this work, we address the problem of low-rank matrix recovery from its under-sampled projections. The recovery is formulated as a Schatten-p norm minimization problem. We proposed a novel algorithm to solve the Schatten-p norm minimization problem based on the FOCUSS (FOCally Under-determined System Solver) approach. We compared our proposed method with state-of-the-art solvers. Experimental evaluation was carried out on two problems – matrix completion and image inpainting. For matrix completion, our proposed method showed better recovery rate than other methods. In the image inpainting problem, our method yields 1.5 dB improvement over the nearest competing algorithm.

***Index Terms*—** Low rank matrix recovery, Matrix Completion, Schatten-p norm

1. INTRODUCTION

Recently there has been considerable interest in the recovery of rank deficient matrices from their under-sampled projections. The practical applicability of such techniques arise in various problems of machine learning and signal processing, such as system identification [1], image inpainting [2], dynamic MRI reconstruction [3], etc. In order to recover a rank deficient matrix, one ideally solves,

$$\min \; rank(X) \text{ subject to } y = A(X) \quad (1)$$

where $X_{n_1 \times n_2}$ is the rank deficient matrix to be solved, $y_{m\times 1}$ is the vector of under-sampled projections and $A:\mathbb{C}^{n_1\times n_2} \rightarrow \mathbb{C}^m$ represents the linear operator. For image inpainting and matrix completion problems like collaborative filtering, system identification and direction of arrival estimation, the operator A is a binary mask. For the problem of dynamic MRI reconstruction, A is a restricted Fourier transform operator.

Rank minimization is an NP hard problem [4]. Theoretical studies in low-rank matrix recovery [4],suggest substituting the NP hard rank minimization problem by its closest convex surrogate – nuclear norm minimization (2). The problem then becomes

$$\min \; \|X\|_* \text{ subject to } y = A(X) \quad (2)$$

where $\|X\|_*$ is the sum of singular values of X.

This is a convex optimization problem that can be solved by semi-definite programming. This problem however requires an increase in the number of samples necessary to arrive at the solution, as compared to the NP hard problem. A list of efficient solvers for (2) can be found in [5].

Most of the theoretical results on low rank matrix recovery have been proven for the nuclear norm minimization. However, it has been seen empirically that non-convex surrogates of the rank of a matrix can yield better recovery results. In [6], a non-convex log-det heuristic was proposed:

$$\min \log\det(X + \eta I) \text{ subject to } y = A(X) \quad (3)$$

where η is a regularization parameter.
Another non-convex heuristic, the reweighted nuclear norm (4), has also been proposed [7].

$$\min \|W_k X W_k\|_* \text{ subject to } y = A(X) \quad (4)$$

where in each iteration, W_k is updated as $W_k = (X_{k-1} + \eta I)^{-1/2}$, η being a regularization parameter.
Yet another non-convex heuristic is the Iterated Re-weighted Least Squares (IRLS) method [8, 9].

$$\min \|W_k^{1/2} X\|_F^2 \text{ subject to } y = A(X) \quad (5)$$

where in each iteration W_k is updated as $W_k = (X_{k-1} X_{k-1}^T)^{-1/2}$.

All past studies in non-convex methods for low-rank matrix recovery [6-9] have shown that non-convex surrogates yield better reconstruction results than the convex nuclear norm. A similar conclusion was corroborated in our previous work [10, 11] where we have proposed solving the low-rank matrix recovery problem by the non-convex Schatten-p norm minimization (6). In these studies the following problem was solved by modified iterative soft thresholding.;

$$\min \|X\|_{S_p}^p \text{ subject to } y = A(X) \quad (6)$$

where $\|X\|_{S_p} = \left(\sum (\sigma_i)^p\right)^{1/p}$

In the present paper, we derive a new algorithm to solve the Schatten-p norm minimization problem by a *FOCally Under-determined System Solver* (FOCUSS) [12] based approach. In the past the FOCUSS approach has been successfully used in developing algorithms for sparse [12], group-sparse [13] and joint-sparse [14] recovery in Compressed Sensing.

In this paper, we test the proposed algorithm on two practical examples. The first problem is that of matrix completion and the second is of image inpainting from images with missing pixels.

The derivation of the algorithm is given in Section 2. Experimental evaluation on simulated and real data is provided in Section 3. Finally in Section 4, the conclusions of the work are discussed.

2. DERIVATION OF THE ALGORITHM

Our problem is to solve the Schatten-p norm minimization problem (6). The derivation is based on the FOCUSS approach [12]. For this work, we will define the Schatten-p norm of the matrix X as $\|X\|_{S_p}^p = Tr(X^T X)^{p/2}$; therefore the problem to be solved is,

$$\min Tr(X^T X)^{p/2} \text{ subject to } y = Ax,\ x = vec(X) \quad (7)$$

The unconstrained Lagrangian form for (7) is,

$$L(x,\lambda) = Tr(X^T X)^{p/2} + \lambda^T (y - Ax) \quad (8)$$

where λ is the vector of Lagrangian multipliers.
The conditions for stationary points of (8) are,

$$\nabla_x L(X,\lambda) = p(XX^T)^{\frac{p}{2}-1} X + A^T \lambda = 0 \quad (9a)$$

$$\nabla_\lambda L(X,\lambda) = Ax - y = 0 \quad (9b)$$

Now (9) can be expressed as,

$$pWx + A^T \lambda = 0,\ W = I \otimes (XX^T)^{\frac{p}{2}-1} \quad (10)$$

where $\otimes$ denotes the Kronecker product.
Solving, x from (10),

$$x = -\frac{1}{p} W^{-1} A^T \lambda \quad (11)$$

W is a block diagonal matrix with positive semi-definite blocks along the diagonal. The problem is that, since W is positive semi-definite, the solution is not numerically stable. Such a problem was encountered while using FOCUSS for sparse signal recovery in Compressed Sensing [15]; in order to get a stable solution W must be positive definite and hence must be regularized. Following [15] and other works [7-9], we propose the following regularization,

$$W_k = I \otimes (X_{k-1} X_{k-1}^T + \varepsilon I)^{\frac{p}{2}-1} \quad (12)$$

Here ε is a small constant that regularizes the solution. This regularization also guarantees that W (and hence W^{-1}) to be positive definite. As $\varepsilon \to 0$, one arrives at the desired solution.

Substituting the value of x from (11) into (9b) and solving for λ we get,

$$\lambda = -p(AW^{-1}A^T)^{-1} y \quad (13)$$

Substituting, the value of λ back in (11), we get,

$$x = W^{-1} A^T (AW^{-1} A^T)^{-1} y \quad (14)$$

In order to efficiently compute x in each iteration, we re-write (14) as,

$$x = R\tilde{x},\ \text{where } \tilde{x} = (AR)^T ((AR)(AR)^T)^{-1} y \quad (15)$$

Here R is the Cholesky decomposition of W^{-1}. The decomposition exists since W^{-1} is a positive definite matrix. The reason, we expressed (14) in the current form (15) is because, $\tilde{x}$ can be solved very efficiently using the LSQR algorithm. Based, on this modification, we propose the following efficient algorithm to solve the Schatten-p norm minimization problem.

Intitialize: $x_0 = A^T (AA^T)^{-1} y$ which is a least squares solution; define ε
Repeat until stopping criterion is met:
Compute: $W_k = I \otimes (X_{k-1} X_{k-1}^T + \varepsilon I)^{\frac{p}{2}-1}$ and $R_k R_k^T = W_k^{-1}$.
Update: $\tilde{x}_k = (AR_k)^T ((AR_k)(AR_k)^T)^{-1} y$ and $x_k = R\tilde{x}_k$.
Reshape x_k to matrix form X_k.
Decrease: $\varepsilon = \varepsilon / 10$ iff $\|x_k - x_{k-1}\|_2 \le tol$

We use two stopping criteria. The first one limits the maximum number of iterations to a certain value. The second stopping criterion is based on the value of the change in the objective function in two consecutive iterations; therefore if the change is nominal, then the iterations stop assuming the solution has reached a local minimum. The update step of the algorithm is solved by LSQR which runs for a maximum of 20 iterations. The value of ε is initialized at 1. The tolerance level for deciding the decrease of ε is fixed at 10^{-3}.

Owing to limitations in space, we are unable to analyze the convergence rate of our proposed algorithm. Since this is a FOCUSS based algorithm, the convergence rate can be shown to super-linear (2-p), by following an analysis similar to [12].

3. EXPERIMENTAL RESULTS

The problem of recovering rank deficient matrices from partially sampled entries arises in various different engineering problems. In this work, we study two well known problems where this is directly applied – i) matrix completion, and ii) image inpainting.

3.1 Matrix Completion

The problem in matrix completion is to recover a low-rank matrix from its partially sampled entries. Such a problem arises in a variety of scenarios including collaborative filtering, system identification and direction of arrival estimation. In this case, the operator A is a binary mask.

First we empirically study the convergence of our algorithm. As mentioned before, being a FOCUSS based method it has a super-linear convergence rate of *(2-p)*. We test the convergence of our algorithm for two cases: *p=1* and *p=0.1*. The size of the matrix is 50 x 60 and has a rank of 5. 40% of the entries were randomly sampled. The decrease of the normalised objective function versus the number of iterations is shown in Fig. 1. The graph is plotted for an average of 100 trials. It is difficult to experimentally verify the exact convergence rates, but it can be seen how fast the algorithm converges for *p=1* as compared to *p=0.1*.

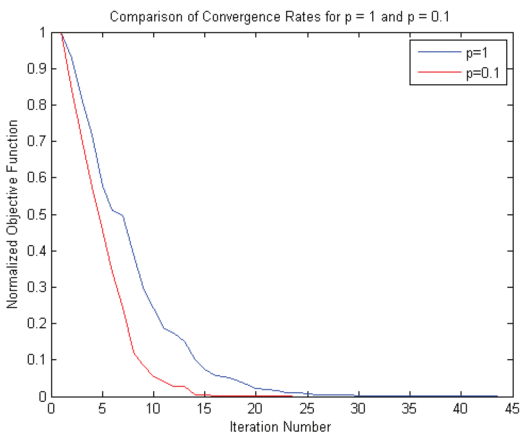

Figure 1. Convergence of Objective Function for p = 1 and p = 0.1

We compare the reconstruction accuracy from our method with the IRLS algorithm [8] and Soft Thresholding method [10]. For all the experiments the size of the matrix is fixed at 50 x 60 and the rank is varied from 1 to 20. 40% of the entries were randomly sampled. The reconstruction is considered a success if the relative mean squared error is less than 10^{-3}. The results are averaged for 100 trials for each value of rank.

Fig.2 shows the graph for a decreasing success rate with an increase in the rank of the matrix. It shows that our proposed FOCUSS based method has the highest success rate, followed by soft thresholding. The IRLS method has the lowest success rate. Comparing Fig. 2a and Fig. 2b, we see how the success rate improves as one moves from *p = 1* to *p = 0.1*.

3.2 Image Inpainting

In image inpainting the problem is to recover missing pixel values. In [2], natural images were modeled as rank deficient matrices and hence the inpainting problem was formulated as a low rank matrix recovery problem and was solved via nuclear norm minimization. In this paper, we experiment with the images of Lena and Barbara. 50% of the pixel values are randomly masked. The images are reconstructed from the masked samples via nuclear norm minimization (p=1). For quantitative evaluation we provide the SNR values in Table 1. The superiority of our proposed method is discernible from the numerical results. Our method gives about 1.5 dB increase over the nearest competing method.

The qualitative evaluation for inpainting can be seen in Fig.3. It shows how the reconstruction artifacts are visibly reduced with our proposed FOCUSS based reconstruction, compared to IRLS and soft thresholding.

Table 1. SNR in dB from different reconstruction algorithms

Image Name	IRLS	Soft-Thresh	Pproposed
Lena	25.7	27.6	**29.1**
Barbara	24.6	26.9	**28.5**

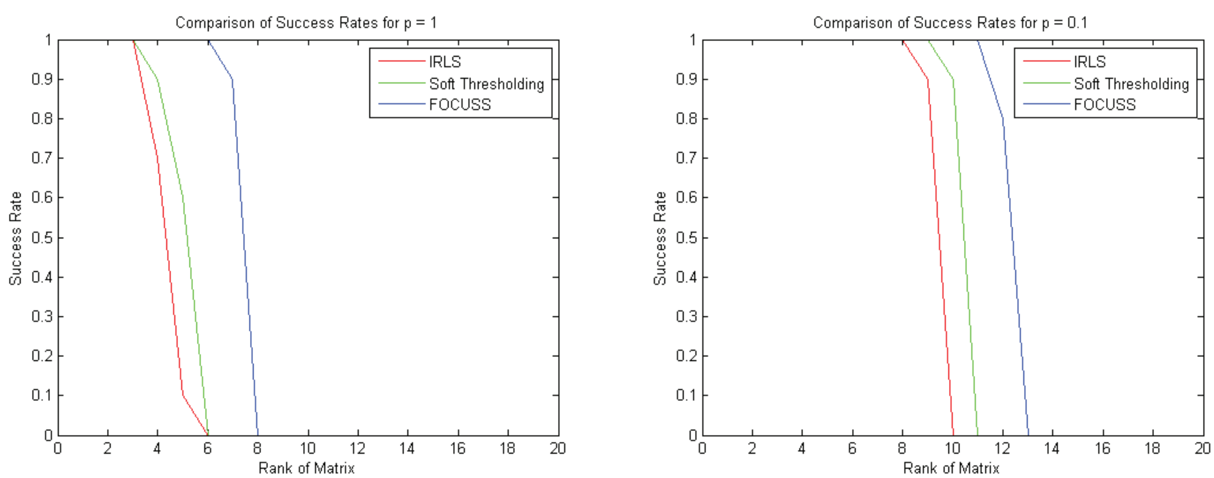

Figure 2. Comparison of Success Rates for Different Algorithms: Left – p = 1 and Right – p = 0.1

Figure3. Lena (top and Barbara (bottom): Left to right – Masked Image, IRLS, Soft Thresholding and Proposed FOCUSS Reconstruction

4. CONCLUSION

In this paper we propose a new algorithm for Schatten-p norm minimization. It is based on the FOCUSS approach [12]. The algorithm has a superlinear convergence rate of 2-p. The convergence of the algorithm has been tested numerically for matrix completion problems. The proposed algorithm was compared against the Iterative Reweighted Least Squares (IRLS) method [8] and the soft thresholding method [10] for Schatten-p norm minimization. The results on matrix completion show that the success rate for low-rank matrix recovery from our proposed algorithm is significantly higher than the other methods compared against. Also the results in image inpainting indicate that our proposed method is able to recover missing pixel values with a higher degree of accuracy among the three competing methods. Our method gives about 1.5 dB increase in SNR over the nearest competing method – this is a significant improvement since in most image processing problems an improvement in 0.5 dB is considered noticeable.

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